On deformations of Eisenstein cohomology classes and applications

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G reductive gp

p prime

$G/_{\mathbb{Q}_p}$ split.

p-adic variations of cohomological automorphic rep'n of $G(\mathbb{A})$.

$K_f \subseteq G(\mathbb{A}_f)$

$G_\infty = G(\mathbb{R})$

$\cup$

Z_∞, $\tilde{M}$ = local system on $S_G(K_f) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty Z_\infty K_f$

$H_c^\bullet(S_G(K_f), \tilde{M})$ . . .

M = alg. rep'n.

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$\lambda \in X(T)^+$ $\quad W_\lambda$ irred alg. rep'n of G of highest wt λ.

$\chi \in \mathfrak{X}(\mathbb{Q}_p)$ $\quad \mathcal{A}_\chi$ = locally analy. induction $\mathrm{Ind}_{B^- \cap I}^{I} \chi$

λ analytic $\quad I$ = Iwahori subgp

$\mathcal{D}_\chi$ = continuous dual of $\mathcal{A}_\chi$.

$$H^\bullet(S_G(K_f), W_\lambda^\vee) \leftarrow H^\bullet(S_G(K_f), \mathcal{D}_\lambda)$$

(See Stevens.)

$$= H^\bullet_{cusp} \oplus H^\bullet_{Eis}$$

$$H^\bullet_{cusp} = \bigoplus_{\pi \text{ irred cusp}} \pi_f^{K_f} \otimes H^\bullet(\mathrm{Lie}\, G_\infty, K_\infty, \pi_\infty \otimes W_\lambda^\vee)$$

$$H^\bullet_{Eis} = \bigoplus_{\pi \text{ Eis} \ldots} \quad \ldots$$

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Emerton: $\varprojlim_{m} \varinjlim_{K_f} H^{\bullet}(S_G(K_f), \mathbb{Z}/p^n\mathbb{Z})$

Ash-Stevens: look at $H^{\bullet}(S_G(K_f^p I), \mathcal{D}_\lambda)$

Relate these using derived categories ...

...

Eigen varieties:

S: finite set of primes containing all the ones at which K_f^p is ramified.

$T^{++} \subset T^{+} = \{ t \in T(\mathbb{Q}_p) \mid t^{-1} N(\mathbb{Z}_p) t \subseteq N(\mathbb{Z}_p) \}$

$T^{\#} = \{ \quad " \quad \subsetneq \quad " \quad \}$

$R_{S,p} = \mathcal{C}_c^{\infty}(G(\mathbb{A}_f^{p,S}) /\!/ K_f^{p,S}, \mathbb{Z}) \otimes \mathbb{Z}[T^{+}]$

$(f^p \otimes t \to f^p \otimes 1_{I_n t I_n})$

$\circlearrowright$

$H^{\bullet}(S_G(K_f I_n), \mathcal{D}_\lambda)$ or $W_\lambda^{\vee}$

$\mathcal{E}_{K_f^p, S} = \{ x = (\theta, \lambda)$ where $\lambda \in \mathcal{X}(\bar{\mathbb{Q}}_p)$

$\theta : R_{S,p} \longrightarrow \bar{\mathbb{Q}}_p$
occurring in $H^{\bullet}(S_G(K_f^p I_n), \mathcal{D}_\lambda)$
and such that θ finite slope $\theta(t) \neq 0 \ \forall t \in T^{+} \}$

locally rigid analytic ...

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Eisenstein cohomology :

$G \supset P$ max. parabolic.
$\quad = M_P N_P$.

$$H^\bullet_{\mathrm{Eis}}(K_f, W_\lambda^\vee) = \bigoplus_{\substack{\sigma \text{ cuspidal} \\ \text{rep}^n \text{ of } M(\mathbb{A}) \\ \sigma = \sigma_f \otimes \sigma_\infty}} \left(\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \sigma\right)^{K_f^p} \otimes H^\bullet(\mathrm{Lie}\, M_\infty, K_\infty, \sigma_\infty \otimes \underbrace{H^\bullet(\mathrm{Lie}\, N_P, W_\lambda^\vee)}_{\text{can be computed.}})$$

λ regular.

$$\bigoplus_{w \in W^M = W/W_M} W^M_{w * \lambda}$$

alg. irred. rep'n of M of highest wt. $w(\lambda+\rho)-\rho$.

π cohomological automorphic rep'n.

$\pi = \pi_f \otimes \pi_\infty \qquad \pi_\infty$ of wt. λ

to make the p-adic variation, one needs to make a choice

$\pi_p^I \big|_{\mathbb{Z}_p[T^+]} \leftarrow$ choose a character θ occurring in π_p

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If $\pi_p = \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi \qquad \chi$ character of $T(\mathbb{Q}_p)$.

χ unramified, $\qquad \pi_p^I$ has dim. $= \# W$.
$\qquad\qquad = \bigoplus_{w \in W} w * \chi$ as a rep'n of T^+.

This choice determines the type of deformation.

e.g. $GL_2(\mathbb{Q})$, f level N prime to p $\to f_\alpha, f_\beta$ of level $\Gamma_0(p) \cap \Gamma_1(N)$. where α, β are roots of the Hecke char. polynomial of f.

In the case of ES the choice is very important

Ex. E_k wt k level 1.

$$= \frac{L(1-k)}{2} + \sum_{n=1}^\infty \sigma_{k-1}(n) q^n$$

$E_k^{\mathrm{ord}}(q) \rightsquigarrow$ eigenvalue of U_p is $1 \rightsquigarrow$ slope $= 0 \xrightarrow{\text{deform}}$ family of Eisenstein

$E_k^{ord}(q)$ ⇝ eigenvalue of U_p is 1 ⇝ slope $= 0$ $\xrightarrow{\text{deform}}$ family of Eisenstein series.

$E_h^{asp}(q)$ ⇝ ———— p^{h-1} ⇝ slope $h-1$ ⟶ family which is generically cuspidal.

Example: σ cohomological cusp. rep'n of $GL_2(K)$, K imag. quad fld.

$G = U(2,2)$ quasi-split unitary gp of signature $(2,2)$.

$P \subset G$ max. proper parabolic $P = MN$

$$P = \begin{array}{c c} & \begin{array}{cc} 2 & 2 \end{array} \\ \begin{array}{c} 2 \\ 2 \end{array} & \left(\begin{array}{c|c} * & * \\ \hline & * \end{array}\right) \end{array} \qquad M \cong GL_2(K) : \begin{pmatrix} g & \\ & {}^c g^{-1} \end{pmatrix}$$

Consider Θ a p-stabilization of $\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \sigma_f$.

Assume that the weight of σ is regular.

$$H^\bullet_{\text{?}p}(\quad)[\Theta] = \left(\mathrm{Ind}_{P(\mathbb{A}_f^{(p)})}^{G(\mathbb{A}_f^{(p)})} \sigma_f^{(p)}\right)^{K_f^{(p)}} \otimes \left(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \sigma_p\right)^I$$

for large degrees.

$$\otimes H^\bullet(\mathfrak{gl}(2,\mathbb{C}), U(2); \underbrace{H^\bullet(\mathrm{Lie}\,N, W_\lambda^\vee)}_{\text{decompose this} \ldots})$$

$\bigoplus_{w \in W^M_{Eis}} W^M_{w \star \lambda}$

E_C Eisenstein class in $H^\bullet(S_G(K_f I), W_\lambda)[\Theta]$ λ regular

Can apply the previous result on eigenvariety to construct deformation of Θ.

If Galois rep'n of σ exists, say ρ_σ, then the Galois rep'n attached to Θ is

$$\rho_\Theta \cong \rho_\sigma \oplus \check{\rho}_\sigma \circ (\text{cx conj})$$

$$(\rho_\sigma : G_K \to GL_2(\bar{\mathbb{Q}}_p)\,.)$$

If Θ is "far" from being ordinary, we should expect to have a generically cuspidal family.

But: $H^i(\mathfrak{gl}(2,\mathbb{C}), \ldots) \neq 0$ for $i = 1, 2$

$\Rightarrow$ $m(K_f^p I, \lambda, \Theta) = 0$

$\Rightarrow$ the deformation is not of full dimension

(So we cannot apply the thm) But we expect it to be of codim 1, because θ is going to occur in 2 consecutive degrees.

However, there is no reason to think that the pts in this family are classical, therefore, we need to find information regarding Galois rep'ns for this pt. in order to apply this strategy.

Galois Representations:

If π is cusp. cohomological for G = unitary gp. $\quad G = GU(r,s)$, $r+s=n$

we expect (Book project) $\rightsquigarrow$ $R_p(\pi): G_K \longrightarrow GL(n, \bar{\mathbb{Q}}_p)$.

Question: What about any pt $x = (\Theta, \lambda) \in \mathcal{E}_{K^p_f, S}$?

If $m(K_f, \theta, \lambda) \neq 0$, should be able to attach Galois repn. (by p-adic approx.)

o/w, we do not know. (No clear expectations on strategy.)

Conj If $x \in \mathcal{E}_{K^p_f, S}$, then there should exist

$R_p(x): G_K \rightarrow GL_n(\bar{\mathbb{Q}}_p)$

attached to x.

Refine: $R_p(x)$ exists and $R_p(x)\big|_{G_{\mathbb{Q}_p}}$ is a triangular rep ((φ, Γ) - module attached to this is triangular)

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(via Shimura variety)

Can define a Galois action on $H^{i_\theta}(S_G(K_f), \mathcal{F}_x)[\theta] = R_p(x, \Lambda^r)$

If $R_p(x)$ exists, $R_p(x, \Lambda^r)$ should be $= \Lambda^r R_p(x)$.

Congruence relation on $R_p(x, \Lambda^r)$

Conj. Let $M(x, \Lambda^r)$ be the (φ, Γ) module attached to $R_p(x, \Lambda^r)|_{G_{\mathbb{Q}_p}}$

Then $M(x, \Lambda^r)$ is triangular, with character $\delta_{j,r}$ among the set of characters $\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_r}$ $i_1 < i_2 < \ldots < i_r$.

Conj $\Rightarrow$ if θ is "far" from being ordinary, then $R_p(x, \Lambda^r)$ needs to be big enough...

If we deform the Eisenstein class in a very non-ordinary way...

expect to get a big family of Galois rep'n $\to \rho_0$.